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LETTER TO THE EDITOR

Hilbert space condition on conserved quantities in second-order discrete time classical mechanics

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Abstract. Kowalski's method of mapping first-order discrete time systems to Hilbert space is discussed in the context of conserved quantities in second-order discrete time mechanics.

In this letter we show that a necessary and sufficient condition for any first-order† quantity C to be conserved in second-order discrete time mechanics is that the corresponding operator $\hat{C}$ commutes with the appropriate generalization of Kowalski's evolution operator $\hat{M}$, originally defined for first-order discrete time systems [1]. Without loss of generality, we shall limit the discussion to a single real dynamical variable x_n indexed by an integer n, obeying the second-order equation of motion

$$x_{n+1} \cong f(x_n, x_{n-1}) \qquad 0 < n < N \tag{1}$$

where the symbol $\cong$ denotes an equality holding on a dynamical (i.e. actual) path. We will suppose that equation (1) may be obtained by an application of Cadzow's variational principle [2] to the action sum

$$A_N = \sum_{n=1}^{N} F^n \tag{2}$$

where the first-order system function $F^n := F(x_n, x_{n-1})$ is the discrete time analogue of a Lagrangian of the form $L := L(x, \dot{x})$ in continuous time mechanics. This gives the second-order equation of motion

$$\frac{\partial}{\partial x_n}\{F^n + F^{n+1}\} \cong 0 \qquad 0 < n < N \tag{3}$$

which we assume may be solved to give x_{n+1} explicitly as in (1).

Our interest here is in first-order conserved quantities $C^n := C(x_n, x_{n-1})$, which by definition satisfy the condition $C^{n+1} \cong C^n$. We may construct such quantities by applying Logan's variant of Noether's theorem [3] as follows. Choose some infinitesimal transformation of the coordinates of the form $\delta x_n = \epsilon u_n$, where ϵ is infinitesimal and $u_n := u(x_n, x_{n-1})$ and then

$$\delta F^n \cong \epsilon \left\{ \frac{\partial F^n}{\partial x_n} u_n - \frac{\partial F^{n-1}}{\partial x_{x-1}} u_{n-1} \right\}. \tag{4}$$

† By definition, if x_n is a variable indexed by a discrete parameter n then a pth-order function g has $p+1$ arguments $x_r, x_{r+1}, \ldots, x_{r+p}$ for some integer r.

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If we can show that δF^n is of the form $\epsilon(v_n - v_{n-1})$, where $v_n := v(x_n, x_{n-1})$ is some function other than $\frac{\partial F_n}{\partial x_n} u_n$, then we can immediately deduce that the quantity $C^n := v_n - \frac{\partial F_n}{\partial x_n} u_n$ is conserved modulo the equations of motion (1) or (3).

To use Kowalski's Hilbert space description of classical discrete time systems, we first rewrite our second-order equation of motion (1) as a pair of first-order equations. If we define $y_n := x_{n-1}$ then we may write

$$x_{n+1} \cong f(x_n, y_n) \qquad y_{n+1} \cong g(x_n, y_n) \qquad 0 < 1 < N \tag{5}$$

where $g(x_n, y_n) := x_n$.

Kowalski's method is to associate a creation and annihilation operator with each dynamical degree of freedom. We associate the operators $\hat{a}, \hat{a}^+$ with the x_n variable and the operators $\hat{b}, \hat{b}^+$ with the y_n variable. The important non-zero commutators are

$$[\hat{a}, \hat{a}^+] = [\hat{b}, \hat{b}^+] = 1. \tag{6}$$

Following Kowalski, we define the Hilbert space states

$$|x, y; n\rangle := \exp\{\tfrac{1}{2}({x_n}^2 + {y_n}^2 - {x_1}^2 - {y_1}^2)\}|x_n, y_n\rangle \qquad 0 < n \leqslant N \tag{7}$$

where $|x_n, y_n\rangle$ is the normalized coherent state

$$|x_n, y_n\rangle := \exp\{-\tfrac{1}{2}({x_n}^2 + {y_n}^2 + x_n\hat{a}^+ + y_n\hat{b}^+)\}|0\rangle. \tag{8}$$

The appropriate generalization of Kowalski's evolution operator is

$$\hat{M} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\hat{a}^+)^k}{k!}\frac{(\hat{b}^+)^l}{l!}[f(\hat{a}, \hat{b}) - \hat{a}]^k[g(\hat{a}, \hat{b}) - \hat{b}]^l \tag{9}$$

and then we may readily show that

$$|x, y; n+1\rangle \cong \hat{M}|x, y; n\rangle \qquad 0 < n < N. \tag{10}$$

To see how the constants of motion are encoded into the Hilbert space description, we note the basic property of the coherent states, i.e.

$$\hat{a}|x, y; n\rangle = x_n|x, y; n\rangle \qquad \hat{b}|x, y; n\rangle = y_n|x, y; n\rangle. \tag{11}$$

Given any classical first-order function $C(x, y)$ we define its associated operator $\hat{C}$ by $\hat{C} := C(\hat{a}, \hat{b})$. Then, assuming C is an analytic function of its arguments, we readily deduce

$$\hat{C}|x, y; n\rangle = C(x_n, y_n)|x, y; n\rangle = C^n|x, y; n\rangle. \tag{12}$$

Hence we find

$$[\hat{C}, \hat{M}]|x, y; n\rangle = (C^{n+1} - C^n)|x, y; n+1\rangle. \tag{13}$$

If now we know C is conserved, i.e. $C^{n+1} \cong C^n$, then we may use the overcompleteness property of the states $|x, y; n\rangle$ to deduce

$$[\hat{C}, \hat{M}] = 0 \tag{14}$$

which is the required necessary and sufficient condition satisfied by any operator function of $\hat{a}$ and $\hat{b}$ representing a conserved first-order quantity in the original classical system. This is analogous to the commutation of conserved operators with the temporal evolution operator $\hat{U}(t)$ in continuous time quantum mechanics. We note that this result holds despite the fact that $\hat{C}$ is not Hermitian.

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